Quantum Expanders and Geometry of Operator Spaces II

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November 7, 2012

In this appendix to [5] we give a quick proof of an inequality that can be substituted to Hastings's result from [2], quoted as Lemma 1.9 in [5]. Our inequality is less sharp but also appears to apply with more general (and even matricial) coefficients. It shows that up to a universal constant all moments of the norm of a linear combination of the form

$$S = \sum\nolimits_j a_j U_j \otimes \bar{U}_j (1 - P)$$

are dominated by those of the corresponding Gaussian sum

$$S' = \sum\nolimits_j a_j Y_j \otimes \bar{Y}'_j.$$

The advantage is that S' is now simply separately a Gaussian random variable with respect to the independent Gaussian random matrices (Y_j) and (Y'_j).
We recall that we denote by P the orthogonal projection onto the orthogonal of the identity. Also recall we denote by S_2^N the space M_N equipped with the Hilbert-Schmidt norm (S_2^N can also be naturally identified with $\ell_2^N \otimes_2 \overline{\ell_2^N}$). We will view elements of the form $\sum x_j \otimes \bar{y}_j$ with $x_j, y_j \in M_N$ as linear operators acting on S_2^N as follows

$$T(\xi) = \sum\nolimits_j x_j \xi y_j^*,$$

so that

$$\|\sum x_j \otimes \bar{y}_j\| = \|T\|_{B(S_2^N)}. \tag{0.1}$$

We denote by (U_j) a sequence of i.i.d. random $N \times N$-matrices uniformly distributed over the unitary group $U(N)$. We will denote by (Y_j) a sequence of i.i.d. Gaussian random $N \times N$-matrices, more precisely each Y_j is distributed like the variable Y that is such that $\{Y(i,j)N^{1/2}\}$ is a standard family of N^2 independent complex Gaussian variables with mean zero and variance 1. In other words $Y(i,j) = (2N)^{-1/2}(g_{ij} + \sqrt{-1}g'_{ij})$ where g_{ij}, g'_{ij} are independent Gaussian normal $N(0,1)$ random variables.

We denote by (Y'_j) an independent copy of (Y_j).

We will denote by $\|.\|_q$ the Schatten q-norm ($1 \le q \le \infty$), i.e. $\|x\|_q = (\mathrm{tr}(|x|^q))^{1/q}$, with the usual convention that for $q = \infty$ this is the operator norm.

Lemma 0.1. *There is an absolute constant C such that for any $p \geq 1$ we have for any scalar sequence (a_j) and any $1 \leq q \leq \infty$*

$$\mathbb{E}\|\sum_1^n a_j U_j \otimes \bar{U}_j(1-P)\|_q^p \leq C^p \mathbb{E}\|\sum_1^n a_j Y_j \otimes \bar{Y}'_j\|_q^p,$$

(in fact this holds for all k and all matrices $a_j \in M_k$ with $a_j \otimes$ in place of a_j).

Proof. We assume that all three sequences (U_j), (Y_j) and (Y'_j) are mutually independent. The proof is based on the well known fact that the sequence (Y_j) has the same distribution as $U_j|Y_j|$, or equivalently that the two factors in the polar decomposition $Y_j = U_j|Y_j|$ of Y_j are mutually independent. Let $\mathcal{E}$ denote the conditional expectation operator with respect to the σ-algebra generated by (U_j). Then we have $U_j \mathbb{E}|Y_j| = \mathcal{E}(U_j|Y_j|) = \mathcal{E}(Y_j)$, and moreover

$$(U_j \otimes \bar{U}_j)\mathbb{E}(|Y_j| \otimes \overline{|Y_j|}) = \mathcal{E}(U_j|Y_j| \otimes \overline{U_j|Y_j|}) = \mathcal{E}(Y_j \otimes \overline{Y_j}).$$

Let

$$T = \mathbb{E}(|Y_j| \otimes \overline{|Y_j|}) = \mathbb{E}(|Y| \otimes \overline{|Y|}).$$

Then we have

$$\sum a_j(U_j \otimes \bar{U}_j)T(I-P) = \mathcal{E}((\sum a_j Y_j \otimes \overline{Y_j})(I-P)).$$

Note that by rotational invariance of the Gaussian measure we have $(U \otimes \bar{U})T(U^* \otimes \bar{U}^*) = T$. Indeed since UYU^* and Y have the same distribution it follows that also $UYU^* \otimes \overline{UYU^*}$ and $Y \otimes \bar{Y}$ have the same distribution, and hence so do their modulus.
Viewing T as a linear map on $S_2^N = \ell_2^N \otimes \overline{\ell_2^N}$, this yields

$$\forall U \in U(N) \quad T(U\xi U^*) = UT(\xi)U^*.$$

Representation theory shows that T must be simply a linear combination of P and $I-P$. Indeed, the unitary representation $U \mapsto U \otimes \bar{U}$ on $U(N)$ decomposes into exactly two distinct irreducibles, by restricting either to the subspace $\mathbb{C}I$ or its orthogonal. Thus, by Schur's Lemma we know a priori that there are two scalars χ'_N, χ_N such that $T = \chi'_N P + \chi_N(I-P)$. We may also observe $\mathbb{E}(|Y|^2) = I$ so that $T(I) = I$ and hence $\chi'_N = 1$, therefore

$$T = P + \chi_N(I-P).$$

Moreover, since $T(I) = I$ and T is self-adjoint, T commutes with P and hence $T(I-P) = (I-P)T$, so that we have

$$\sum_1^n a_j(U_j \otimes \bar{U}_j)(1-P)T = \mathcal{E}\sum_1^n a_j(Y_j \otimes \bar{Y}_j)(I-P). \tag{0.2}$$

We claim that T is invertible and that there is an absolute constant C so that

$$\|T^{-1}\| = {\chi_N}^{-1} \leq C.$$

From this and (0.2) follows immediately that for any $p \geq 1$

$$\mathbb{E}\|\sum_1^n a_j(U_j \otimes \bar{U}_j)(1-P)\|_q^p \leq C^p \mathbb{E}\|\sum_1^n a_j(Y_j \otimes \bar{Y}_j)(1-P)\|_q^p. \tag{0.3}$$

To check the claim it suffices to compute χ_N. For $i \neq j$ we have a priori $T(e_{ij}) = e_{ij}\langle T(e_{ij}), e_{ij}\rangle$ but (since $\mathrm{tr}(e_{ij}) = 0$) we know $T(e_{ij}) = \chi_N e_{ij}$. Therefore for any $i \neq j$ we have $\chi_N = \langle T(e_{ij}), e_{ij}\rangle$, and the latter we can compute

$$\langle T(e_{ij}), e_{ij}\rangle = \mathbb{E}\mathrm{tr}(|Y|e_{ij}|Y|^* e_{ij}^*) = \mathbb{E}(|Y|_{ii}|Y|_{jj}).$$

Therefore,

$$N(N-1)\chi_N = \sum_{i\neq j} \mathbb{E}(|Y|_{ii}|Y|_{jj}) = \sum_{i,j} \mathbb{E}(|Y|_{ii}|Y|_{jj}) - \sum_j \mathbb{E}(|Y|_{jj}^2) = \mathbb{E}(\mathrm{tr}|Y|)^2 - N\mathbb{E}(|Y|_{11}^2).$$

Note that $\mathbb{E}(|Y|_{11}^2) = \mathbb{E}\langle |Y|e_1, e_1\rangle^2 \leq \mathbb{E}\langle |Y|^2 e_1, e_1\rangle = \mathbb{E}\|Y(e_1)\|_2^2 = 1$, and hence

$$N(N-1)\chi_N = \sum_{i\neq j} \mathbb{E}(|Y|_{ii}|Y|_{jj}) \geq \mathbb{E}(\mathrm{tr}|Y|)^2 - N.$$

Now it is well known that $E|Y| = b_N I$ where b_N is determined by $b_N = N^{-1}\mathbb{E}\mathrm{tr}|Y| = N^{-1}\|Y\|_1$ and $\inf_N b_N > 0$ (see e.g. [3, p.80]). Actually, by Wigner's limit theorem, when $N \to \infty$, $N^{-1}\|Y\|_1$ tends almost surely to $\tau|c_1|$. Therefore, $N^{-2}\mathbb{E}(\mathrm{tr}|Y|)^2$ tends to $(\tau|c_1|)^2$. We have

$$\chi_N = (N(N-1))^{-1} \sum_{i\neq j} \mathbb{E}(|Y|_{ii}|Y|_{jj}) \geq (N(N-1))^{-1}\mathbb{E}(\mathrm{tr}|Y|)^2 - (N-1)^{-1},$$

and this implies

$$\liminf_{N\to\infty} \chi_N \geq (\tau|c_1|)^2.$$

In any case, we have

$$\inf_N \chi_N > 0,$$

proving our claim.

We will now deduce from (0.3) the desired estimate by a classical decoupling argument for multilinear expressions in Gaussian variables.
We first observe $\mathbb{E}((Y \otimes \bar{Y})(I-P)) = 0$. Indeed, by orthogonality, a simple calculation shows that $\mathbb{E}(Y \otimes \bar{Y}) = \sum_{ij} \mathbb{E}(Y_{ij}\overline{Y_{ij}}) e_{ij} \otimes \overline{e_{ij}} = \sum_{ij} N^{-1} e_{ij} \otimes \overline{e_{ij}} = P$, and hence $\mathbb{E}((Y \otimes \bar{Y})(I-P)) = 0$.

We will use

$$(Y_j, Y_j') \overset{dist}{=} ((Y_j + Y_j')/\sqrt{2}, (Y_j - Y_j')/\sqrt{2})$$

and if $\mathbb{E}_Y$ denotes the conditional expectation with respect to Y we have (recall $\mathbb{E}(Y_j \otimes \bar{Y}_j)(I-P) = 0$)

$$\sum_1^n a_j Y_j \otimes \bar{Y}_j(I-P) = \mathbb{E}_Y(\sum_1^n a_j Y_j \otimes \bar{Y}_j(I-P) - \sum_1^n a_j Y_j' \otimes \bar{Y}_j'(I-P)).$$

Therefore

$$\mathbb{E}\|\sum_1^n a_j Y_j \otimes \bar{Y}_j(1-P)\|_q^p \leq \mathbb{E}\|\sum_1^n a_j Y_j \otimes \bar{Y}_j(1-P) - \sum_1^n a_j Y_j' \otimes \bar{Y}_j'(I-P))\|_q^p$$

$$= \mathbb{E}\|\sum_1^n a_j(Y_j + Y_j')/\sqrt{2} \otimes \overline{(Y_j + Y_j')/\sqrt{2}}(1-P) - \sum_1^n a_j(Y_j - Y_j')/\sqrt{2} \otimes \overline{(Y_j - Y_j')/\sqrt{2}}(I-P))\|_q^p$$

$$= \mathbb{E}\|\sum_1^n a_j(Y_j \otimes \overline{Y'_j} + Y'_j \otimes \overline{Y_j})(1-P)\|_q^p$$

and hence by the triangle inequality

$$\leq 2^p \mathbb{E}\|\sum_1^n a_j(Y_j \otimes \overline{Y'_j})(1-P)\|_q^p.$$

Thus we conclude a fortiori

$$\mathbb{E}\|\sum_1^n a_j U_j \otimes \bar{U}_j(1-P)\|_q^p \leq (2C)^p \mathbb{E}\|\sum_1^n a_j(Y_j \otimes \overline{Y'_j})\|_q^p.$$

□

Theorem 0.2. *Let C be as in the preceding Lemma. Let*

$$\hat{S}^{(N)} = \sum_1^n a_j U_j \otimes \bar{U}_j(1-P).$$

Then

$$\limsup_{N\to\infty} \mathbb{E}\|\hat{S}^{(N)}\| \leq 4C(\sum |a_j|^2)^{1/2}. \tag{0.4}$$

Moreover we have almost surely

$$\limsup_{N\to\infty} \|\hat{S}^{(N)}\| \leq 4C(\sum |a_j|^2)^{1/2}. \tag{0.5}$$

Proof. A very direct argument is indicated in Remark 0.4 below, but we prefer to base the proof on [1] in the style of [5] in order to make clear that it remains valid with matrix coefficients. By [5, (3.1)] applied twice (for $k=1$) (see also Remark 3.5 in [5]) one finds for any even integer p

$$E\mathrm{tr}|\sum_1^n a_j(Y_j \otimes \overline{Y'_j})|^p \leq (\mathbb{E}\mathrm{tr}|Y|^p)^2(\sum |a_j|^2)^{p/2} \tag{0.6}$$

Therefore by the preceding Lemma

$$E\mathrm{tr}|\hat{S}^{(N)}|^p \leq C^p(\mathbb{E}\mathrm{tr}|Y|^p)^2(\sum |a_j|^2)^{p/2},$$

and hence a fortiori

$$E\|\hat{S}^{(N)}\|^p \leq N^2 C^p(\mathbb{E}\|Y\|^p)^2(\sum |a_j|^2)^{p/2}.$$

We then complete the proof, as in [5], using only the concentration of the variable $\|Y\|$. We have an absolute constant β' and $\varepsilon(N) > 0$ tending to zero when $N \to \infty$, such that

$$(\mathbb{E}\|Y\|^p)^{1/p} \leq 2 + \varepsilon(N) + \beta'\sqrt{p/N},$$

and hence

$$(E\|\hat{S}^{(N)}\|^p)^{1/p} \leq N^{2/p} C(2 + \varepsilon(N) + \beta'\sqrt{p/N})^2(\sum |a_j|^2)^{1/2}.$$

Fix $\varepsilon > 0$ and choose p so that $N^{2/p} = \exp \varepsilon$, i.e. $p = 2\varepsilon^{-1} \log N$ (note that this is ≥ 2 when N is large enough) we obtain

$$E\|\hat{S}^{(N)}\| \leq (E\|\hat{S}^{(N)}\|^p)^{1/p} \leq 4e^{\varepsilon} C(1 + \varepsilon^{-1}\varepsilon'(N))(\sum |a_j|^2)^{1/2}$$

where $\varepsilon'(N) \to 0$ when $N \to \infty$, and (0.4) follows.
Let $R_N = 4C(1+\varepsilon^{-1}\varepsilon'(N))(\sum |a_j|^2)^{1/2}$. By Tshebyshev's inequality $(E\|\hat{S}^{(N)}\|^p)^{1/p} \leq e^{\varepsilon} R_N$ implies

$$\mathbb{P}\{\|\hat{S}^{(N)}\| > e^{2\varepsilon} R_N\} \leq \exp -\varepsilon p = N^2.$$

From this it is immediate that almost surely

$$\limsup_{N\to\infty} \|\hat{S}^{(N)}\| \leq e^{2\varepsilon} 4C(\sum |a_j|^2)^{1/2}$$

and hence (0.5) follows. $\square$

Remark 0.3. The same argument can be applied when $a_j \in M_k$ for any integer $k > 1$. Then we find

$$\limsup_{N\to\infty} \mathbb{E}\| \sum_1^n a_j \otimes U_j \otimes \bar{U}_j (1 - P)\| \leq 4C \max\{\| \sum a_j^* a_j\|^{1/2}, \| \sum a_j a_j^*\|^{1/2}\}.$$

Moreover we have almost surely

$$\limsup_{N\to\infty} \| \sum_1^n a_j \otimes U_j \otimes \bar{U}_j (1 - P)\| \leq 4C \max\{\| \sum a_j^* a_j\|^{1/2}, \| \sum a_j a_j^*\|^{1/2}\}.$$

Remark 0.4. In the case of scalar coefficients a_j the proof extends also to double sums of the form

$$\sum_{ij} a_{ij} U_i \otimes \bar{U}_j (I - P).$$

We refer the reader to [4, Theorem 16.6] for a self-contained proof of (0.6) for such double sums.

Acknowledgment. I am grateful to S. Szarek for a useful suggestion.

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